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A flat plate in a rotating, stratified flow

M. R. FOSTER
Department of Aeronautical and Astronautical Engineering, The Ohio State University, Columbus, Ohio 43210, USA
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SUMMARY
The boundary layer over a flat plate of semi-infinite extent in a stratified and rotating flow grows forward from the trailing edge, and is characterized by an intrinsic length scale L, which represents the distance from the trailing edge at which vortex stretching becomes just as important in the boundary layer as baroclinic vorticity production. Near the trailing edge, the layer is essentially the layer in a purely stratified flow; far upstream (many L), it is an Ekman layer. The boundary layer entrains no fluid, but induces at its edge a transverse velocity component which drives an higher-order streamwise outer flow. If the flow is bounded above and below by horizontal planes, the Wiener-Hopf solution for this outer flow indicates that the disturbance decays rapidly downstream, but the transverse velocity component is non-zero far upstream.

1. Introduction

The long forward-facing wake that occurs in front of an obstacle in a slow stratified fluid flow has been studied for two decades, much of the early work having been done by Long (c.f. [10], for example), and has since been detailed in particular problems by Graebel [7], Janowitz [8], and Foster [6]. In particular, when the obstacle is a plate aligned with the flow, a forward-growing boundary layer occurs. The structure of that boundary layer on a semi-infinite plate was given by Martin and Long [11] in 1968. Subsequently, Brown [3] gave the solution for the development of that boundary layer into an upstream wake for the case when the plate has finite length. Since then, very little has been done to extend these results to include effects of Coriolis force when the fluid is also rotating at an angular velocity, say, Ω. Redekopp [13] has delineated the variety of boundary layers that may occur on such a plate depending on the relative sizes of the Reynolds number, Froude number, and Rossby number.

We consider here a semi-infinite plate lying in the plane $y = 0, x < 0$. The plate is in motion to the left, say, parallel to itself at speed U. The fluid in which the plate moves is taken to be incompressible and non-diffusive. We suppose that, where the fluid is undisturbed by the motion of the plate, its density is given by $\rho = 1-\beta y, \beta > 0$. The motion occurs in a frame rotating at angular velocity Ω, normal to the plate. If $-g\nabla y$ is the acceleration of gravity and ν the kinematic viscosity of the fluid, there are four dimensional quantities that characterize the motion in the boundary layer: $\Omega, \sqrt{g\beta}, U, \nu$. (We suppose that any horizontal boundaries are far enough away to leave the boundary layer itself unaffected to first order. We leave to Sec. 5 what detailed restrictions this places on the parameter range in which this solution is valid.)

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$\Omega/\sqrt{g\beta}$ is a measure of the relative importance of rotation to stratification. The only other parameter that arises naturally in the equations is

$$\epsilon = \frac{\Omega^3 U^2}{\nu(g\beta)^2}$$

which can be written as $[U(U/\Omega)/\nu] \times \Omega^4/(g\beta)^2$ and hence is proportional to a Reynolds number based on an inertial length scale U/Ω. In this work, we obtain the structure of the flow for $\epsilon \to 0$. Since there is no plate length scale, an intrinsic scale

$$L \equiv \frac{\nu g\beta}{\Omega^2 U} \tag{1.2}$$

is the appropriate length. The parameter ϵ, above, is the square of the ratio of the Ekman layer thickness and L.

We find that the boundary layer grows forward from the trailing edge in a manner predicted by Martin and Long [11]. However, far upstream (several L), the boundary layer is an Ekman layer that carries no net mass in the streamwise direction. In fact, all along its length, the boundary layer entrains no fluid from the inviscid flow (see Redekopp [13] for a brief discussion of this feature), but a cross-stream velocity of order U is induced at its edge by vortex stretching. That non-zero lateral velocity is related to an $O(\epsilon U)$ stream-wise flow, and satisfies a certain third order parabolic equation, whose solution is given by the Wiener-Hopf technique.

The higher order outer flow decays rapidly in the downstream direction from the plate; however, in the upstream direction, we find that the $O(U)$ transverse velocity component does not go to zero. In fact, very far upstream, the transverse component is found to vary linearly with y, from $-U$ on the plate to zero on the horizontal planes.

2. Formulation and outer expansion

The semi-infinite plate mentioned in Sec. 1 is taken to occupy $y = 0, \mathfrak{X} < 0$ at $t = 0$; it is translating at speed U toward negative $\mathfrak{X}$. The fluid through which the plate is moving is, as mentioned previously, incompressible, non-diffusive, and stratified, and otherwise at rest; it is bounded above and below by planes $y = H_T$ and $y = -H_B$ respectively. We choose a coordinate system translating with the plate, so $x = \mathfrak{X} + Ut$. In that frame the flow is steady. The Boussinesq equations of motion are

$$\nabla \cdot \mathbf{u} = 0, \tag{2.1}$$

$$(\mathbf{u} \cdot \nabla)\mathbf{u} + 2\mathbf{\Omega} \times \mathbf{u} + \nabla p = \nu\nabla^2\mathbf{u} - \rho g\mathbf{j}, \tag{2.2}$$

$$\mathbf{u} \cdot \nabla\rho = 0 \tag{2.3}$$

where (u,v,w) are the components of $\mathbf{u}$ in the Cartesian directions (x,y,z). The rotation vector is normal to the plate, the y-direction; in (2.2), $\mathbf{j} = \nabla y$. The boundary conditions are

$$\mathbf{u} = \mathbf{0} \text{ on } y = 0, x < 0, \tag{2.4}$$

$$\mathbf{u} = \nabla(Ux) \text{ on } y = H_T \text{ and } y = -H_B, \tag{2.5}$$

and, far from the plate where the fluid is undisturbed,

$$\begin{aligned} &\mathbf{u} \sim \nabla(Ux), \\ &\rho \sim 1 - \beta y, \beta > 0. \end{aligned} \tag{2.6}$$

Equation (2.3) admits the solution, in terms of the stream function ψ,

$$\rho = 1 - \beta\psi/U \tag{2.7}$$

which is consistent with (2.6), but leaves aside, as usual in these problems (cf. Graebel [7], Janowitz [8], Foster [6]), the question of what function of ψ is correct for ρ in regions of closed streamlines. The stream function in (2.7) is such that

$$u = \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x}. \tag{2.8}$$

Let $\psi = UL\tilde{\psi}$, $w = U\tilde{w}$, and put $p = 2\Omega Uz - gy + (\nu^2(g\beta)^3/\Omega^4 U^2)\tilde{p}$. Non-dimensionalizing all lengths by L, and using (2.7) and (2.8), (2.2) becomes

$$\epsilon^2 \frac{g\beta}{\Omega^2} (\tilde{\mathbf{u}} \cdot \tilde{\nabla})\tilde{\mathbf{u}} + 2\epsilon\mathbf{j} \times (\tilde{\mathbf{u}} - \mathbf{i}) + \tilde{\nabla}\tilde{p} = \epsilon^2\tilde{\nabla}^2\tilde{\mathbf{u}} + \mathbf{j}\tilde{\psi}, \tag{2.9}$$

where $\mathbf{i} = \nabla x$. We note that the non-linear terms in (2.9) are apparently negligible only for

$$g\beta/\Omega^2 \ll 1. \tag{2.10}$$

Dropping the $(\tilde{\ })$ notation and neglecting the inertia, and supposing the flow is two-dimensional so nothing depends on z, we have, for (2.9),

$$2\epsilon\mathbf{j} \times (\mathbf{u} - \mathbf{i}) + \nabla p = \epsilon^2\nabla^2\mathbf{u} + \psi\mathbf{j}. \tag{2.11}$$

Elimination of p in (2.11) leads to the equations

$$\frac{\partial\psi}{\partial x} = 2\epsilon\frac{\partial w}{\partial y} + \epsilon^2\nabla^4\psi, \tag{2.12}$$

$$-2\left(\frac{\partial\psi}{\partial y} - 1\right) = \epsilon\nabla^2 w. \tag{2.13}$$

In this paper, we seek solution to (2.12) and (2.13) for $\epsilon \to 0$, subject to non-dimensional versions of (2.4)-(2.6). We suppose that the outer expansions for ψ and w begin

$$\psi = y + \epsilon^a \psi_1 + \ldots, \qquad w = \epsilon^b w_1 + \ldots \tag{2.14}$$

where $a > 0$ and $b \geqslant 0$; the first term in (2.14a) satisfies conditions (2.5) and (2.6a) as well as (2.13), but fails to satisfy no-slip, (2.4). Substitution of (2.14) into (2.12) and (2.13) gives

$$-\frac{\partial \psi_1}{\partial x} = 2\epsilon^{b+1-a}\frac{\partial w_1}{\partial y}, \qquad -\frac{\partial \psi_1}{\partial y} = \tfrac{1}{2}\epsilon^{b+1-a}\nabla^2 w_1. \tag{2.15}$$

Now, if $b + 1 - a > 0$, $\psi_1 = 0$ is the solution of the limit forms of (2.15), which contradicts the assumed form of (2.14). For $b + 1 - a < 0$, (2.15) leads to $w_1 \equiv 0$; so the *only* possibility is $b + 1 - a = 0$. Then, (2.15) reduces to

$$\nabla^2 \frac{\partial w_1}{\partial x} = 4\frac{\partial^2 w_1}{\partial y^2}. \tag{2.16}$$

The boundary condition for w_1 on $y = 0$, $x < 0$, must be deduced by matching with the boundary layer of Sec. 3. We return, in Sec. 4, to the solution of (2.16).

3. The boundary layer

Writing $\psi = \epsilon^{1/2}\Psi$ and $y = \epsilon^{1/2}\zeta$ and putting these into (2.12), (2.13), on letting $\epsilon \to 0$, gives

$$\frac{\partial}{\partial \zeta}\left[\frac{\partial^4 \Psi}{\partial \zeta^4} - \frac{\partial \Psi}{\partial x} + 4\Psi\right] = 4 \tag{3.1}$$

and

$$2\left(\frac{\partial \Psi}{\partial \zeta} - 1\right) = \frac{\partial^2 w}{\partial \zeta^2}. \tag{3.2}$$

Solution of (3.1) may be written as

$$\Psi = \zeta + g(x) + \Phi(x,\zeta) \tag{3.3}$$

where Φ satisfies

$$\frac{\partial^4 \Phi}{\partial \zeta^4} - \frac{\partial \Phi}{\partial x} + 4\Phi = 0, \tag{3.4}$$

and $g(x)$, a result of the integration, is not yet known. Now $\partial\Phi/\partial\zeta \to 0$ for $\zeta \to \infty$ allows (3.3) to match to (2.14), to first order. If Φ is exponentially small for ζ large, as we may verify *a posteriori*, then (3.2) may be integrated to give

$$w = 2 \int_0^{\zeta} \Phi(x,\bar{\zeta})d\bar{\zeta}, \tag{3.5}$$

so that $w = O(1)$ at the boundary-layer edge. So, matching w in (3.5) and (2.14), the outer expansion, requires that $b \equiv 0$. Then, $a \equiv 1$, and matching ψ with (3.3) and (2.14) requires $g(x) \equiv 0$, i.e., there is no vertical velocity at the boundary-layer edge. This lack of entrainment for such boundary layers has previously been noted by both Barcilon and Pedlosky [2] and Redekopp [13].

Before proceeding to the solution, notice that much about the structure of this boundary layer is evident from (3.4). At large values of $-x$, the second term in (3.4) is negligible and we have the Ekman layer equation. For $-x$ small, the second term dominates the third, and we get a layer that grows like $(-x)^{1/4}$; in fact, the solution there is exactly that given by Martin and Long [11]. So, the boundary layer determined by a balance between viscous diffusion and baroclinic vorticity production at small $-x$ undergoes a transition far upstream to an Ekman layer, where viscous diffusion balances vortex stretching.

We write $x = -\xi/4$, and proceed to the solution of

$$\frac{\partial^4 \Phi}{\partial \zeta^4} + 4 \frac{\partial \Phi}{\partial \xi} + 4\Phi = 0, \tag{3.6}$$

with

$$\Phi(\xi,\infty) = 0, \tag{3.7}$$

and

$$\Phi(\xi,0) = 0, \qquad \frac{\partial \Phi}{\partial \zeta}(\xi,0) = -1, \tag{3.8}$$

the last of which guarantees satisfaction of no-slip, (2.4). Solution is by Laplace transformation in ξ, and is

$$\Phi = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \hat{\Phi}(s,\zeta) e^{s\xi} ds, \qquad c > 0, \tag{3.9}$$

where

$$\hat{\Phi}(s,\zeta) = -\frac{1}{\gamma s} e^{-\gamma\zeta} \sin(\gamma\zeta), \qquad \gamma \equiv (1+s)^{1/4}. \tag{3.10}$$

The horizontal velocity, on taking a ζ derivative of (3.10) and inverting, is given by

$$u = 1 + \sqrt{2} \sin\left(\zeta - \frac{\pi}{4}\right) e^{-\zeta} - \frac{e^{-\xi}}{\sqrt{2}\,\pi} I(\eta,\xi) \tag{3.11}$$

where η is the similarity variable, $\sqrt{2}\zeta/\xi^{1/4}$, and

$$I(\eta,\xi) \equiv \int_0^{\infty} \frac{e^{-u}}{u+\xi} \left[\sin\left(\frac{\pi}{4}\right) e^{-\eta u^{1/4}} - \sin\left(\frac{\pi}{4} + \eta u^{1/4}\right) \right] du. \tag{3.12}$$

In addition, the integral of (3.10) over ζ indicates, from (3.5), that

$$w \sim -\operatorname{erf}(\sqrt{\xi}), \quad \zeta \to \infty. \tag{3.13}$$

The first two terms in (3.11) are due to a residue at $s = 0$ and constitute the Ekman Layer solution; for $\xi \to \infty$, the final term is small and the layer is an Ekman layer. The term involving $I(\eta,\xi)$ is the modification of the Ekman layer due to the flow stratification. We show, in Figure 1, the solution (3.11) plotted versus the physical variable ζ and also the similarity variable, η, for various values of ξ. The results were obtained by Simpson's rule integration of (3.12). For $\xi = 0$, our solution, (3.11)-(3.12), agrees with the numerical results of Martin and Long [11].

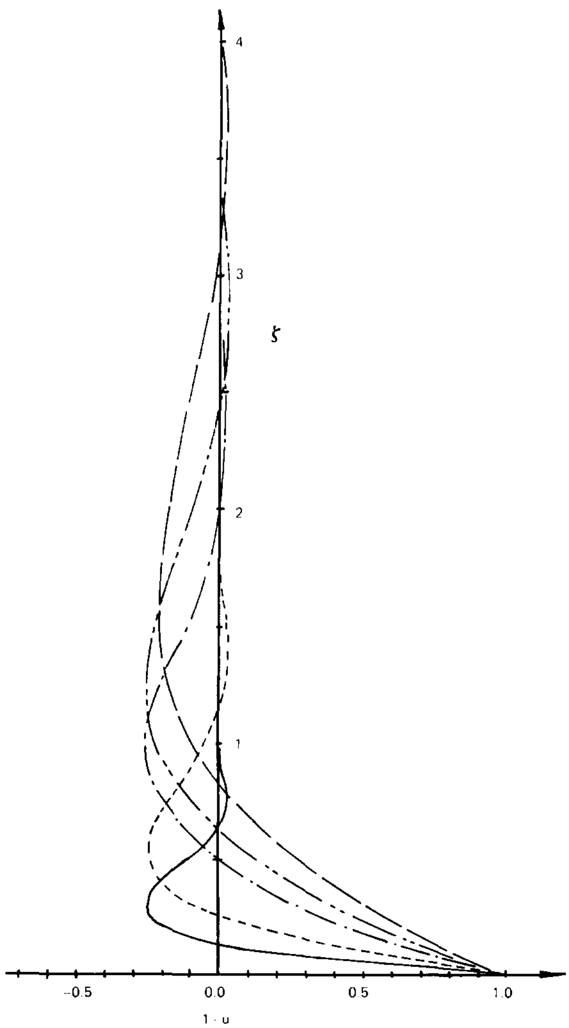

Figure 1(a). Non-dimensional horizontal velocity in the boundary layer versus ζ, for different values of ξ. For ——, $\xi = .001$; for --, $\xi = .01$; for —-—, $\xi = .1$; for —--—, $\xi = 1$; and for —— —— ——, $\xi = 10$.

Asymptotic behavior

We seek the approximate form of (3.11) for large values of η. In evaluating I in (3.12) for $\eta \to \infty$, we note the comment of Erdelyi [5] that the contribution of a finite end-point is more important than the contribution near critical points. The contribution near the lower limit of (3.12) for $\eta \to \infty$ is

$$2\pi e^{-\zeta}\sin(\zeta - \pi/4).$$

No critical points of the part of (3.12) with the factor $\exp(-\eta u^{1/4})$ lie in $|\,arg\,(u)\,| < \pi$, so it makes no contribution in the steepest descent calculation. The second term, with the sine

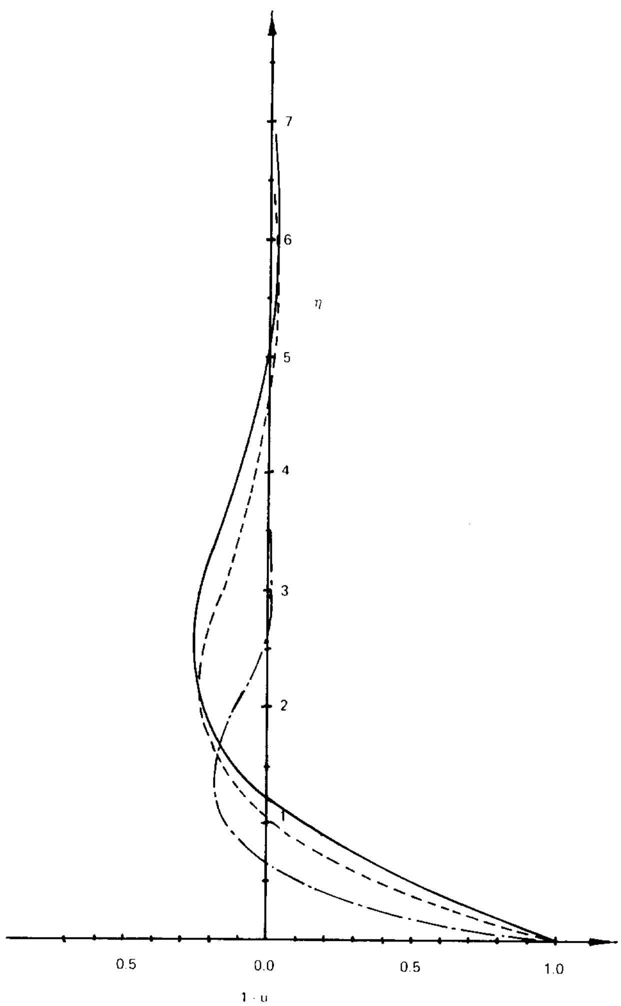

Figure 1(b). Non-dimensional horizontal velocity in the boundary layer versus the similarity variable $\eta = \sqrt{2\zeta/\xi^{1/4}}$ for various values of ξ. For —— - —— -, ξ = .001, .01, .1 (They are indistinguishable on the graph.); for —, ξ = 1; and for ——; ξ = 10.

function, has a critical point at $u = (\eta/4)^{4/3} e^{-i2\pi/3}$, and the steepest descent method (Erdelyi [5]) leads to

$$u \sim 1 + (1 - e^{-\xi}) \sin(\zeta - \pi/4)e^{-\zeta}$$
$$+ \frac{2e^{-\xi}}{\sqrt{3}\pi} (4/\eta)^{2/3} \frac{P(\eta,\xi/\zeta)e^{-K\eta^{4/3}}}{(4(\xi/\eta)^{4/3} - 1/2)^2 + 3/4},$$
$$P(\eta,\lambda) = \sin(K\eta^{4/3} - \pi/12) + 4\lambda^{4/3} \sin(K\eta^{4/3} + 7\pi/12),$$
$$K \equiv 3^{3/2}/2^{11/3}, \quad \text{for } \eta \to \infty, \tag{3.14}$$

where we have allowed ξ/ζ to have any order whatever. For $\xi \to 0$, we get

$$u \sim 1 + \frac{2}{\sqrt{3}\pi} (4/\eta)^{2/3} e^{-K\eta^{4/3}} \sin(K\eta^{4/3} - \pi/12), \tag{3.15}$$

which is the asymptotic form of the solution of Martin and Long [11]. For $\xi \to \infty$, we obtain the large-η modification to the Ekman layer solution,

$$u \sim 1 + (1 - e^{-\xi})\sin(\zeta - \pi/4) + \frac{1}{\sqrt{3}\pi} \frac{e^{-\xi}}{\xi} e^{-K\eta^{4/3}} \sin(K\eta^{4/3} + 7\pi/12). \tag{3.16}$$

This result is valid for $\xi \gg \zeta \gg \xi^{1/4}$, as $\xi \to \infty$.

4. Higher order outer flow

We now proceed to the solution of (2.16). Matching the boundary-layer solution, (3.4), (3.9), and (3.10), to the expansion (2.14) with, as determined in Sec. 3, $a = 1$, $b = 0$, gives the boundary condition for w_1,

$$w_1(x,0) = -\operatorname{erf}(2\sqrt{-x}), \quad x < 0. \tag{4.1}$$

Clearly, symmetry requires

$$\frac{\partial w_1}{\partial y}(x,0) = 0, \quad x > 0. \tag{4.2}$$

Also, (2.5) indicates that $w_1 = 0$ on $y = +h_T$, and $-h_B$, where $h_T = H_T/L$ and $h_B = H_B/L$. For convenience, we write $x = +(1/4)x^*$, $y = (1/4)y^*$. Then, (2.16) becomes

$$\nabla^{*2} \frac{\partial w_1}{\partial x^*} = \frac{\partial^2 w_1}{\partial y^{*2}}, \tag{4.3}$$

subject to (4.1) and (4.2) in these variables,

$$w_1(x^*,0) = -\operatorname{erf}(\sqrt{-x^*}), \quad x^* < 0, \tag{4.4}$$

and

$$\frac{\partial w_1}{\partial y^*}(x^*,0) = 0, \quad x^* > 0. \tag{4.5}$$

We proceed using the Wiener-Hopf method (cf. Noble [12], Carrier, Krook, and Pearson [4], p. 376 *ff*). In order to guarantee convergent integrals, we must modify (4.4) to

$$w_1^*(x^*,0) = -e^{\delta x^*}\mathrm{erf}(\sqrt{-x^*}), \quad x^* < 0, \tag{4.6}$$

and w^* satisfies (4.3) and (4.5) as well. Then, $w_1 = \lim\limits_{\delta\to 0} w_1^*$. As we shall see, the strip of overlap of the 'plus' and 'minus' functions is of width δ. Writing

$$W \equiv \int_{-\infty}^{\infty} w_1^*(x^*,y^*)e^{-ikx^*}dx^*,$$

and transforming (4.3), we can easily obtain the solution for W,

$$W = \begin{cases} C(k)\dfrac{\sinh(\gamma(h_T^* - y^*))}{\sinh(\gamma h_T^*)}, & y^* > 0, \\[2ex] C(k)\dfrac{\sinh(\gamma(h_B^* + y^*))}{\sinh(\gamma h_B^*)}, & y^* < 0, \end{cases} \tag{4.7}$$

$$\gamma \equiv \left(\frac{k^3}{k+i}\right)^{1/2}. \tag{4.8}$$

Branches are chosen such that $-\frac{3\pi}{2} < \arg(k) \leq \pi/2$ and $3\pi/2 < \arg(k+i) \leq -\pi/2$. Transforming (4.6) and (4.2) gives

$$-\frac{i}{k+i\delta}\sqrt{\frac{i}{k+i}} + A_- = C, \tag{4.9}$$

$$B_+ = -\frac{C\gamma\sinh(\gamma(h_T^* + h_B^*))}{\sinh(\gamma h_T^*)\sinh(\gamma h_B^*)}, \tag{4.10}$$

where

$$A_- \equiv \int_0^{\infty} w_1^*(x^*,0)e^{-ikx^*}dx^*,$$

$$B_+ \equiv \int_{-\infty}^{0}\left[\frac{\partial w_1^*}{\partial y^*}(x^*,0)\right]e^{-ikx^*}dx^*. \tag{4.11}$$

A_- is analytic in some lower half of the k-plane and B_+ in some upper-half, which hopefully overlap. The first term in (4.9) should actually have the branchpoint at $-(1+\delta)i$; however, it turns out to be unnecessary to retain the δ there, since we are interested in the limit solution for $\delta \to 0$. Eliminating C between (4.9) and (4.10) gives the equation to which the Wiener-Hopf technique is to be applied,

$$-JB_+ = A_- - \frac{i}{k+i\delta} \sqrt{\frac{i}{k+i}}, \tag{4.12}$$

where

$$J = \frac{\sinh(\gamma h_T^*)\sinh(\gamma h_B^*)}{\gamma \sinh(\gamma(h_T^* + h_B^*))}.$$

For the (simpler) case where $h_T \equiv h_B$, J simplifies to

$$J = \frac{1}{2\gamma} \tanh(\gamma h_T^*). \tag{4.13}$$

For shortness, we write h for h_T^* in what follows.

The first task is to split J into the ratio J_+/J_-. That is easily done by treating $\log J$ and expressing the tanh as a ratio of the infinite products for the sinh and cosh. A little algebra and solution of a cubic equation leads to

$$\log J = \log \frac{h}{2} + \sum_{n=1}^{\infty} \log \left[\frac{(1 - k/i\alpha_n)(1 + k/i\beta_n)(1 + k/i\delta_n)}{(1 - k/i\alpha_n')(1 + k/i\beta_n')(1 + k/i\alpha_n')}\right] \tag{4.14}$$

where

$$\alpha_n = \frac{2n\pi}{\sqrt{3}h} \cos(\Phi_n/3), \quad \Phi_n = \cos^{-1}\left(\frac{3\sqrt{3}}{2}\frac{h}{n\pi}\right),$$

$$\beta_n = \frac{2n\pi}{\sqrt{3}h} \cos((\pi + \Phi_n)/3), \tag{4.15}$$

$$\delta_n = \frac{2n\pi}{\sqrt{3}h} \cos((\pi - \Phi_n)/3).$$

The formulae for $(\alpha_n', \beta_n', \delta_n')$ are identical to (4.15) with n replaced by $n-1/2$. So long as $\pi^2/27h^2 > 1$, all of these α, β, and δ values are real. Quite clearly from (4.14) then,

$$J_- = \prod_{n=1}^{\infty} \left(\frac{1 - k/i\alpha_n'}{1 - k/i\alpha_n}\right), \tag{4.16}$$

$$J_+ = \prod_{n=1}^{\infty} \left[\frac{(1 + k/i\beta_n)}{(1 + k/i\beta_n')} \frac{(1 + k/i\delta_n)}{(1 + k/i\delta_n')}\right] e^{h/2}. \tag{4.17}$$

Now $\alpha_n \sim \frac{n\pi}{h} + \frac{1}{2}$ for $n \to \infty$, and simple checks of (4.16) and (4.17) for large n indicate that the infinite products for J_+ and J_- each meet the usual test for convergence of an infinite product (Jeffreys and Jeffreys [9], p. 52 *f*). So, (4.12) is

$$-J_+B_+ = A_-J_- - \frac{i}{k+i\delta}\sqrt{\frac{i}{k+i}}\,J_-. \tag{4.18}$$

The usual integral formula for the additive splitting of the final term in (4.18) into '+' and '–' functions (cf. Carrier, Krook, and Pearson [4], p. 383) gives

$$\frac{i}{k+i\delta}\sqrt{\frac{i}{k+i}}\,J_- = R_+ - S_-, \tag{4.19}$$

$$S_-(k) = \sum_{\ell=1}^{\infty} \frac{Q_\ell(\delta)}{\alpha_\ell + ik}\,\frac{1}{\sqrt{1+\alpha_\ell}} \tag{4.20}$$

where

$$Q_\ell(\delta) \equiv \frac{\alpha_\ell}{\alpha_\ell + \delta}\left(\frac{\alpha_\ell}{\alpha'_\ell} - 1\right)\prod_{\substack{n=1\\ n\neq \ell}}^{\infty}\left(\frac{1-\alpha_\ell/\alpha'_n}{1-\alpha_\ell/\alpha_n}\right). \tag{4.21}$$

Therefore

$$-J_+B_+ + R_+ = A_-J_- + S_- = H(k) \tag{4.22}$$

where H must be analytic in $|k| < \infty$.

We show in the Appendix that both the 'plus' and 'minus' sides of (4.22) vanish for large $|k|$, so that by the Liouville theorem, $H \equiv 0$. Then

$$A_- = -\frac{S_-}{J_-} \tag{4.23}$$

and

$$C = -\frac{S_-}{J_-} - \frac{i}{k+i\delta}\sqrt{\frac{i}{k+i}}\,. \tag{4.24}$$

It is easily verified that this solution does indeed satisfy (4.4) and (4.5) if $\delta \to 0$ after the inversion is performed, provided the inversion path lies in the strip $-\delta < \mathrm{Im}(k) < 0$.

By summing residues in the upper-half plane at $i\alpha'_j$, one can show that the solution is

$$w_1 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{\alpha'_j\,Q'_j\,Q_i\,e^{-4\alpha'_j x}}{(\alpha_i - \alpha'_j)\sqrt{1+\alpha_i}}\cos((j-1/2)\pi y/h), \quad x > 0, \tag{4.25}$$

where Q_i here is $Q_i(0)$ and Q'_j is given by a formula like (4.21) with all $\{\alpha_i\}$ and $\{\alpha'_i\}$ interchanged.

The solution in $x < 0$ is technically complex to obtain, since it involves an integral along the cut from $-i$ to $-i\infty$ as well as poles distributed on either side of that cut. Careful evaluation of all of these contributions leads to the solution,

$$w_1 = \left(\frac{|y|}{h_T} - 1\right) + \frac{e^{4x}}{2\pi} \fint_0^\infty \frac{e^{4rx}}{\sqrt{r}(1+r)} \frac{\sin(4((1+r)^3/r)^{1/2}(h-|y|))}{\sin(4((1+r)^3/r)^{1/2}h)}\, dr$$

$$+ \sum_{j=1}^{\infty} [G(\beta_j) + G(\delta_j)] \sin(j\pi\,|y|/h_T), \quad x < 0, \tag{4.26}$$

where the mark on the integral indicates the Cauchy principle value is to be taken and G is given by

$$G(\lambda) = \left(\frac{(\lambda-1)^3}{\lambda}\right)^{1/2} \frac{e^{4\lambda x}}{J_-(-i\lambda)} \frac{1}{3/2-\lambda} \sum_{\ell=1}^{\infty} \frac{Q_\ell}{\sqrt{1+\alpha_\ell}\,(\alpha_\ell+\lambda)}\,. \tag{4.27}$$

We note from (4.26) that for $x \to -\infty$, w_1 is given approximately by

$$w_1 \sim \frac{|y| - h}{h} \tag{4.28}$$

so there is marked alteration of the upstream flow due to the boundary layer driving. However, downstream, (4.25) makes it clear that $w_1 \to 0$ exponentially fast. Figure 2 shows w_1 on $y = 0$ plotted versus x. Notice that the velocity drops from zero (It vanishes like $x^{1/2} \log x$ for $x \to 0^+$, as shown in the Appendix.) to a peak value of $-.1225$ at $x = .0088$, and strongly decays after that. The calculation was made by truncating the series in (4.27) and the infinite products at 750 terms.

5. Final remarks

One result of the solution of Sec. 4 is that the region of undisturbed flow is *downstream* of the plate, since $w_1 \to 0$ there. However, far upstream, noting (4.26), $\psi_1 \nrightarrow 0$; in fact, by (2.15), $\psi_1 \sim 2x\,\mathrm{sgn}(y)$, so that the density above the plate, far upstream, is $1-\beta(y + 2\epsilon x)$. However, (2.7)

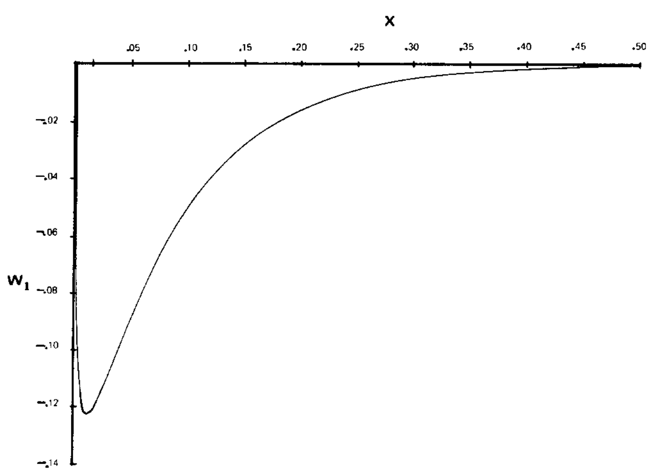

Figure 2. The outer flow transverse velocity component, w_1, downstream of the plate on $y = 0$. The peak value is $-.1225$.

remains valid since all the streamlines in the flow eventually end up downstream, where $\mathbf{u} = \mathbf{i}$. These non-zero vertical velocities of order ϵ on the plate and bounding planes are taken to zero by an order $\epsilon^{1/2}$ correction to the boundary layer solution of Sec. 3.

The 'inertialess' restriction (2.10) may be removed with no alteration in the leading-order boundary-layer structure. The equation for the higher order outer-flow *is* modified, however. Equation (4.3) becomes

$$\nabla^2 \frac{\partial w_1}{\partial x^*} = \frac{\partial^2 w_1}{\partial y^{*2}} + \frac{g\beta}{4\Omega^2} \frac{\partial^2 w_1}{\partial x^{*2}} . \tag{5.1}$$

The solution may proceed along the lines of Sec. 4 by Wiener-Hopf technique, but things are a bit more complex.

Finally, as noted in passing in Sec. 1, we require that the boundary layer be thin compared to the distance between horizontal planes; a sufficient condition is $E \equiv \nu/\Omega H_T^2 \ll 1$. We also require that the non-dimensional h_T of Sec. 4 be $O(1)$. Since $H_T/L = (\epsilon/E)^{1/2}$, we require $E = O(\epsilon)$ to make this analysis valid. Put another way, the Rossby number, $R_0 \equiv U/\Omega H_T$, must be quite small, *viz.*, $R_0 = O\left(\frac{g\beta}{\Omega^2} E\right)$.

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Appendix

In order to complete the Wiener-Hopf procedure, we study the behavior of J_-, A_- and S_-, for $|k| \to \infty$. We proceed with these in sequence.

(i) J_- for $|k| \to \infty$

We found J_- to be given by (4.16),

$$J_-(k) = \prod_{n=1}^{\infty} \left(\frac{1 - k/i\alpha_n}{1 - k/i\alpha_n'} \right) . \tag{A.1}$$

The logarithm is

$$\log J_- = \sum_{n=1}^{\infty} \log \left(\frac{1 - k/i\alpha_n}{1 - k/i\alpha_n'} \right) . \tag{A.2}$$

For a finite value of n in (A.2), $|k| \to \infty$ gives a term of $O(1/k)$. The large-n terms in the series (A.2) determine the large-k form of $\log J_-$. Thus,

$$\log J_- \sim \log \hat{J}_- + O(1/|k|), \quad |k| \to \infty^* \tag{A.3}$$

where

$$\log \hat{J}_- = \sum_{n=1}^{\infty} \log \left(\frac{1 - hk/i(n-1/2)\pi}{1 - hk/in\pi} \right) . \tag{A.4}$$

We have used the asymptotic approximation for α_n for $n \to \infty$, *viz.*, $\alpha_n \sim n\pi/h + 1/2$. For convenience, we write

$$\frac{\hat{J}'_-}{\hat{J}_-} \sim \frac{ih}{2\pi} \sum_{n=1}^{\infty} \frac{1}{(n + ihk/\pi)^2} , \quad |k| \to \infty. \tag{A.5}$$

This sum is easily identified with a sum in Abramowitz and Stegun ([1]), p. 259), related to the psi function, $\Gamma'(z)/\Gamma(z)$. The result is that the integral of (A.5) gives

$$\log \hat{J}_- \sim \tfrac{1}{2} \left[\frac{\Gamma'(ihk/\pi)}{\Gamma(ihk/\pi)} + \frac{\pi}{ihk} \right] \sim \tfrac{1}{2} \log k + O(1/k). \tag{A.6}$$

Therefore, by (A.6) and (A.3),

$$J_- \sim k^{1/2} \text{ for } |k| \to \infty. \tag{A.7}$$

(ii) S_- for $|k| \to \infty$

From (4.20),

$$S_-(k) = \sum_{n=1}^{\infty} \frac{Q_n}{\alpha_n + ik} \frac{1}{\sqrt{1 + \alpha_n}} \tag{A.8}$$

Since the infinite-product Q_n converges for all values of n, we can write

$$|Q_n| \leqslant M < \infty. \tag{A.9}$$

In the region of the plane $\mathrm{Im}(k) < 0$, note that

$$|\alpha_n + ik| = |\alpha_n - \mathrm{Im}(k) + iR\ell(k)| \geqslant |\alpha_n|^{\frac{1}{2}+\delta} |k|^{\frac{1}{2}-\delta} \tag{A.10}$$

for any $\delta > 0$. Then, bounding (A.8),

$$|S_-| \leqslant \frac{M}{|k|^{1/2+\delta}} \sum_{n=1}^{\infty} (\alpha_n)^{-1-\delta}. \tag{A.11}$$

The sum exists for all $\delta > 0$, so $|S_-|$ vanishes faster than $|k|^{-1/2}$ for $|k| \to \infty$ in $\mathrm{Im}(k) < 0$.

(iii) A_- for $|k| \to \infty$

* This result may be obtained rigorously.

For x^* and y^* small, the approximate version of (4.3) is

$$\nabla^2 w_1^* = 0 \tag{A.12}$$

which must satisfy

$$w_1^* = -\frac{2}{\sqrt{\pi}}\sqrt{-x^*} \quad \text{on } y^* = 0, \quad x^* < 0, \tag{A.13}$$

$$\frac{\partial w_1^*}{\partial y} = 0, \quad y^* = 0, \quad x^* > 0.$$

The solution to (A.12) and (A.13) is

$$w_1^* = \frac{2}{\pi^{3/2}} R\ell[(x^* + iy^*)^{1/2} \log (x^* + iy^*)]. \tag{A.14}$$

Now,

$$A_- = \int_0^\infty w_1^*(x^*,0) e^{-ikx^*} dx^*. \tag{A.15}$$

For $|k| \to \infty$ with $\mathrm{Im}(k) < 0$, (A.15) indicates that the small x^* behavior of $w_1^* |_{y=0}$ is important. From (A.14),

$$w_1^* \sim \frac{2}{\pi^{3/2}} x^{*1/2} \log x^* \qquad \text{for } x^* \to 0^+ \text{ on } y^* = 0,$$

so insertion into (A.15) gives

$$A_- \sim \frac{1}{\pi (ik)^{3/2}} \log k. \tag{A.16}$$

Substitution of (A.7), (A.11), and (A.16) into (4.22) for $|k| \to \infty$ indicates clearly that $H \equiv 0$. Similar arguments involving J_+, B_+ and R_+ give the same result, and are not given here for brevity.

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